

# Metric gauge fields in Deformed Special Relativity

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## Abstract

We show that, in the framework of Deformed Special Relativity (DSR), namely a (four-dimensional) generalization of the (local) space-time structure based on an energy-dependent "deformation" of the usual Minkowski geometry, two kinds of gauge symmetries arise, whose spaces either coincide with the deformed Minkowski space or are just internal spaces to it. This is why we named them "*metric gauge theories*". In the case of the internal gauge fields, they are a consequence of the deformed Minkowski space (DMS) possessing the structure of a generalized Lagrange space. Such a geometrical structure allows one to define curvature and torsion in the DMS.

# 1 Introduction

It is well known that gauge theories play presently a basic role in describing all the known interactions. In all cases, gauge symmetries are related to physical fields directly arising from the symmetries ruling some given interaction; on one side, this leads to the rising of a new, dynamical gauge field; on the other hand, if the gauge symmetry is broken, such a circumstance provides one with new — often unforeseen — informations about the structural properties of the interaction considered.

Often, as known as well, in spite of the fact that the physical world is the usual Minkowski space-time, the gauge manifold is *not* the usual, Minkowski one. For instance, in the case of the usual Minkowski space, the gauge symmetry of electrodynamics does actually work in an auxiliary space (Weyl charge space). It is therefore worth to investigate when and where gauge symmetries can be introduced in a Minkowski space, and to lead to significant physical results.

It is just the purpose of the present paper to show that this circumstance occurs in the framework of *Deformed Special Relativity (DSR)*, namely a (four-dimensional) generalization of the (local) space-time structure based on an energy-dependent "deformation" of the usual Minkowski geometry [1, 2]. As we shall see, in DSR two kinds of gauge symmetries arise, whose spaces either coincide with the deformed Minkowski space (DMS)  $\widetilde{M}$  or are just internal spaces to it. This is why we named them "*metric gauge theories*".

The paper is organized as follows. In Sect.2, we review the basic features of DSR that are relevant to our purposes. Sect.3 discuss DSR as a metric gauge theory. Metric gauge fields can be external (Subsect.3.1) or internal (Subsect.3.2). The last topic is related to the structure of DMS as generalized Lagrange space, whose main properties are summarized. Subsect. 3.2.2 deals with the structure of  $\widetilde{M}$  as generalized Lagrange space. The internal gauge fields of  $\widetilde{M}$  are discussed in Subsect.3.2.4. In 3.3 we present a possible experimental evidence for such metric gauge fields. Conclusions and perspectives are given in Sect.4.

## 2 Elements of Deformed Special Relativity

### 2.1 Energy and Geometry

The geometrical structure of the physical world — both at a large and a small scale — has been debated since a long. After Einstein, the generally accepted view considers the arena of physical phenomena as a four-dimensional space-time, endowed with a *global*, curved, Riemannian structure and a *local*, flat, Minkowskian geometry.

However, an analysis of some experimental data concerning physical phenomena ruled by different fundamental interactions have provided evidence for a local departure from Minkowski metric [1, 2]: among them, the lifetime of the

(weakly decaying)  $K_s^0$  meson, the Bose-Einstein correlation in (strong) pion production and the superluminal propagation of electromagnetic waves in waveguides. These phenomena seemingly show a (local) breakdown of Lorentz invariance, together with a plausible inadequacy of the Minkowski metric; on the other hand, they can be interpreted in terms of a deformed Minkowski spacetime, with metric coefficients depending on the energy of the process considered [1, 2].

All the above facts suggested to introduce a (four-dimensional) generalization of the (local) space-time structure based on an energy-dependent "deformation" of the usual Minkowski geometry of  $M$ , whereby the corresponding deformed metrics ensuing from the fit to the experimental data seem to provide an *effective dynamical description of the relevant interactions* (at the energy scale and in the energy range considered).

An analogous energy-dependent metric seems to hold for the gravitational field (at least locally, i.e. in a neighborhood of Earth) when analyzing some classical experimental data concerning the slowing down of clocks.

Let us shortly review the main ideas and results concerning the (four-dimensional) deformed Minkowski spacetime  $\widehat{M}$ .

The four-dimensional "deformed" metric scheme is based on the assumption that spacetime, in a preferred frame which is *fixed* by the scale of energy  $E$ , is endowed with a metric of the form

$$ds^2 = b_0^2(E)c^2dt^2 - b_1^2(E)dx^2 - b_2^2(E)dy^2 - b_3^2(E)dz^2 = g_{DSR\mu\nu}(E)dx^\mu dx^\nu; \\ g_{DSR\mu\nu}(E) = (b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)), \quad (1)$$

with  $x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$ ,  $c$  being the usual speed of light in vacuum. We named "Deformed Special Relativity" (DSR) the relativity theory built up on metric (1).

Metric (1) is supposed to hold locally, i.e. in the spacetime region where the process occurs. It is supposed moreover to play a *dynamical* role, and to provide a geometric description of the interaction considered. In this sense, DSR realizes the so called "*Finzi Principle of Solidarity*" between space-time and phenomena occurring in it <sup>1</sup>(see [3]). Furthermore, we stress that, from

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<sup>1</sup>Let us recall that in 1955 the Italian mathematician Bruno Finzi stated his "*Principle of Solidarity*" (PS), that sounds "*It's (indeed) necessary to consider space-time TO BE SOLIDLY CONNECTED with the physical phenomena occurring in it, so that its features and its very nature do change with the features and the nature of those. In this way not only (as in classical and special-relativistic physics) space-time properties affect phenomena, but reciprocally phenomena do affect space-time properties. One thus recognizes in such an appealing "Principle of Solidarity" between phenomena and space-time that characteristic of mutual dependence between entities, which is peculiar to modern science.*" Moreover, referring to a generic N-dimensional space: "*It can, a priori, be pseudoeuclidean, Riemannian, non-Riemannian. But — he wonders — how is indeed the space-time where physical phenomena take place? Pseudoeuclidean, Riemannian, non-Riemannian, according to their nature, as requested by the principle of solidarity between space-time and phenomena occurring in it.*"

Of course, Finzi's main purpose was to apply such a principle to Einstein's Theory of General Relativity, namely to the class of gravitational phenomena. However, its formulation is as

the physical point of view,  $E$  is the measured energy of the system, and thus a merely phenomenological (non-metric) variable<sup>2</sup>.

We notice explicitly that the spacetime  $\widetilde{M}$  described by (1) is flat (it has zero four-dimensional curvature), so that the geometrical description of the fundamental interactions based on it differs from the general relativistic one (whence the name "deformation" used to characterize such a situation). Although for each interaction the corresponding metric reduces to the Minkowskian one for a suitable value of the energy  $E_0$  (which is characteristic of the interaction considered), the energy of the process is fixed and cannot be changed at will. Thus, in spite of the fact that *formally* it would be possible to recover the usual Minkowski space  $M$  by a suitable change of coordinates (e.g. by a rescaling), this would amount, in such a framework, to be a mere mathematical operation devoid of any physical meaning.

As far as phenomenology is concerned, it is important to recall that a local breakdown of Lorentz invariance may be envisaged for all the four fundamental interactions (electromagnetic, weak, strong and gravitational) whereby *one gets evidence for a departure of the spacetime metric from the Minkowskian one* (in the energy range examined). The explicit functional form of the metric (1) for all the four interactions can be found in [1, 2]. Here, we confine ourselves to recall the following basic features of these energy-dependent phenomenological metrics:

1) Both the electromagnetic and the weak metric show the same functional behavior, namely

$$g_{DSR\mu\nu}(E) = \text{diag} (1, -b^2(E), -b^2(E), -b^2(E)); \quad (2)$$

$$b^2(E) = \begin{cases} (E/E_0)^{1/3}, & 0 \leq E \leq E_0 \\ 1, & E_0 \leq E \end{cases} \quad (3)$$

with the only difference between them being the threshold energy  $E_0$ , i.e. the energy value at which the metric parameters are constant, i.e. the metric becomes Minkowskian; the fits to the experimental data yield

$$E_{0,e.m.} = 5.0 \pm 0.2 \mu eV; \quad E_{0w} = 80.4 \pm 0.2 GeV; \quad (4)$$

2) for strong and gravitational interactions, the metrics read:

$$g_{DSR}(E) = \text{diag} (b_0^2(E), -b_1^2(E), -b_2^2(E), -b_3^2(E)); \quad (5)$$

$$\begin{aligned} b_{0,strong}^2(E) &= b_{3,strong}^2(E) = \begin{cases} 1, & 0 \leq E < E_{0strong} \\ (E/E_{0strong})^2, & E_{0strong} < E \end{cases}; \\ b_{1,strong}^2(E) &= \left(\sqrt{2}/5\right)^2; \quad b_{2,strong}^2 = (2/5)^2; \end{aligned} \quad (6)$$

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general as possible, so to apply in principle to all the known physical interactions. Therefore, Finzi's PS is at the very ground of any attempt at geometrizing physics, i.e. describing physical forces in terms of the geometrical structure of space-time.

<sup>2</sup>As is well known, all the present physically realizable detectors work *via* their electromagnetic interaction in the usual space-time  $M$ . So,  $E$  is the energy of the system measured in *fully Minkowskian conditions*.

$$b_{0,grav}^2(E) = \begin{cases} 1, & 0 \leq E < E_{0grav} \\ \frac{1}{4}(1 + E/E_{0grav})^2, & E_{0grav} < E \end{cases} \quad (6')$$

with

$$E_{0s} = 367.5 \pm 0.4 GeV; E_{0grav} = 20.2 \pm 0.1 \mu eV. \quad (7)$$

Let us stress that, in this case, contrarily to the electromagnetic and the weak ones, *a deformation of the time coordinate occurs*; moreover, *the three-space is anisotropic*<sup>3</sup>, with two spatial parameters constant (but different in value) and the third one variable with energy in an "over-Minkowskian" way (namely it reaches the limit of Minkowskian metric for decreasing values of  $E$ , with  $E > E_0$ ) [1, 2].

As a final remark, we stress that actually *the four-dimensional energy-dependent spacetime  $\widetilde{M}$  is just a manifestation of a larger, five-dimensional space in which energy plays the role of a fifth dimension*. Indeed, it can be shown that the physics of the interaction lies in the curvature of such a five-dimensional spacetime, in which the four-dimensional, deformed Minkowski space is embedded. Moreover, *all* the phenomenological metrics (2), (3) and (5), (6) can be obtained as solutions of the vacuum Einstein equations in this generalized Kaluza-Klein scheme [1, 2].

## 2.2 Field Deformation

We want now to show that the deformation of space-time, expressed by the metric  $g_{DSR}$  (Eq.(1)), does affect also the external fields applied to the physical system considered.

Let us consider for instance the case of a physical process ruled by the electromagnetic interaction. Therefore, the Minkowski space  $M$  is endowed with the electromagnetic tensor  $F_{\mu\nu}(x)$  (external e.m. field) acting on the system. Of course  $F_\nu^\mu(x) = g_{SR}^{\mu\rho} F_{\rho\nu}(x)$ .

In the deformed Minkowski space  $\widetilde{M}$ , the covariant components of the electromagnetic tensor read

$$\widetilde{F}_{\mu\nu} = g_{DSR\mu\rho} F_\nu^\rho = g_{DSR\mu\rho} g_{SR}^{\mu\sigma} F_{\sigma\nu}, \quad (8)$$

where

$$(g_{DSR\mu\rho} g_{SR}^{\mu\sigma}) = \text{diag}(b_0^2, b_1^2, b_2^2, b_3^2) = (b_\sigma^2 \delta_\rho^\sigma). \quad (9)$$

We have therefore

$$\widetilde{F}_{0\nu} = b_0^2 F_{0\nu}; \widetilde{F}_{1\nu} = b_1^2 F_{1\nu}; \widetilde{F}_{2\nu} = b_2^2 F_{2\nu}; \widetilde{F}_{3\nu} = b_3^2 F_{3\nu}, \quad (10)$$

or

$$\widetilde{F}_{\mu\nu} = b_\mu^2 F_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3 \quad (11)$$

(no sum on repeated indices!).

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<sup>3</sup>At least for strong interaction; nothing can be said for the gravitational one.

It follows that the tensor  $\tilde{F}_{\mu\nu}$  is not antisymmetric:

$$\tilde{F}_{\mu\nu} \neq -\tilde{F}_{\nu\mu}. \quad (12)$$

The result shown here for the electromagnetic interaction can be generalized to other fundamental interactions described by tensor fields.

On account of the well-known identification

$$\tilde{F}_{0i} = \tilde{E}_i, \tilde{F}_{12} = -\tilde{B}_3, \tilde{F}_{23} = -\tilde{B}_1, \tilde{F}_{31} = -\tilde{B}_2 \quad (13)$$

(and analogously for  $F_{\mu\nu}$ ), we can write, for the energy density  $\tilde{\mathcal{E}}$  of the deformed electromagnetic field:

$$\tilde{\mathcal{E}} = \frac{\tilde{\mathbf{E}}^2 + \tilde{\mathbf{B}}^2}{8\pi} = \frac{b_0^4 \mathbf{E}^2 + b_1^4 B_3^2 + b_2^4 B_1^2 + b_3^4 B_2^2}{8\pi}, \quad (14)$$

to be compared with the standard expression for the e.m. field  $\mathbf{E}$ ,  $\mathbf{B}$ :

$$\mathcal{E} = \frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi}. \quad (15)$$

*There is therefore a difference in the energy associated to the electromagnetic field in the deformed space-time region.* We have, for the energy density

$$\Delta\mathcal{E} = \mathcal{E} - \tilde{\mathcal{E}}. \quad (16)$$

We can state that the difference  $\Delta\mathcal{E}$  represents *the energy spent by the interaction in order to deform the space-time geometry.*

We can therefore conclude that *the deformation of space-time does affect the field itself that deforms the geometry of the space.* There is therefore a feedback between space and interaction which fully implements the Solidarity Principle.

### 3 DSR as Metric Gauge Theory

#### 3.1 External metric gauge fields

It is clear from the discussion of the phenomenological metrics describing the four fundamental interactions in DSR that the Minkowski space  $M$  is the space-time manifold of background of any experimental measurement and detection (namely, of any process of acquisition of information on physical reality). In particular, we can consider this Minkowski space as that associated to the electromagnetic interaction above the threshold energy  $E_{0,e.m.}$ . Therefore, in modeling the physical phenomena, one has to take into account this fact. The geometrical nature of interactions, *i.e.* assuming the validity of the Finzi principle, means that one has to suitably *gauge* (with reference to  $M$ ) the space-time metrics with respect to the interaction — and/or the phenomenon — under study. In other words, one needs to "adjust" suitably the local metric of space-time according to the interaction acting in the region considered. We can name such a

procedure "*Metric Gaugement Process*" (M.G.P.). Like in usual gauge theories a different phase is chosen in different space-time *points*, in DSR different metrics are associated to different space-time *manifolds* according to the interaction acting therein. We have thus a gauge structure on the space of manifolds

$$\widetilde{\mathcal{M}} \equiv \cup_{g_{DSR} \in \mathcal{P}(E)} \widetilde{M}(g_{DSR}), \quad (17)$$

where  $\mathcal{P}(E)$  is the set of the energy-dependent pseudoeuclidean metrics of the type (1). This is why it is possible to regard Deformed Special Relativity as a *Metric Gauge Theory*. In this case, we can consider the related fields as *external metric gauge fields*.

However, let us notice that DSR can be considered as a metric gauge theory from another point of view, on account of the dependence of the metric coefficients on the energy. Actually, once the MGP has been applied, by selecting the suitable gauge (namely, the suitable *functional form* of the metric) according to the interaction considered (thus implementing the Finzi principle), the metric dependence on the energy implies another different gauge process. Namely, the metric is gauged according to the process under study, thus selecting the *given* metric, with the *given values* of the coefficients, suitable for the given phenomenon.

We have therefore a *double* metric gaugement, according, on one side, to the interaction ruling the physical phenomenon examined, and on the other side to its energy, in which the metric coefficients are the analogous of the gauge functions<sup>4</sup>.

## 3.2 Internal Metric Gauge Fields

We want now to show that the deformed Minkowski space  $\widetilde{M}$  of Deformed Special Relativity does possess another well-defined geometrical structure, besides the deformed metrical one. Precisely, we will show that  $\widetilde{M}$  is a *generalized Lagrange space* [6]. As we shall see, this implies that DSR admits a different, *intrinsic* gauge structure.

### 3.2.1 Deformed Minkowski Space as Generalized Lagrange Space

**Generalized Lagrange Spaces** Let us give the definition of generalized Lagrange space [4], since usually one is not acquainted with it.

Consider a N-dimensional, differentiable manifold  $\mathcal{M}$  and its (N-dimensional) tangent space in a point,  $T\mathcal{M}_{\mathbf{x}}$  ( $\mathbf{x} \in \mathcal{M}$ ). As is well known, the union

$$\bigcup_{\mathbf{x} \in \mathcal{M}} T\mathcal{M}_{\mathbf{x}} \equiv T\mathcal{M} \quad (18)$$

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<sup>4</sup>The analogy of this second kind of metric gauge with the standard, non-abelian gauge theories is more evident in the framework of the five-dimensional space-time  $\mathfrak{R}_5$  (with energy as extra dimension) embedding  $\widetilde{M}$ , on which Deformed Relativity in Five Dimensions (DR5) is based (see [1, 2]). In  $\mathfrak{R}_5$ , in fact, energy is no longer a parametric variable, like in DSR, but plays the role of fifth (metric) coordinate. The invariance under such a metric gauge, not manifest in four dimensions, is instead recovered in the form of the isometries of the five-dimensional space-time-energy manifold  $\mathfrak{R}_5$ .

has a fibre bundle structure. Let us denote by  $\mathbf{y}$  the generic element of  $T\mathcal{M}_{\mathbf{x}}$ , namely a vector tangent to  $\mathcal{M}$  in  $\mathbf{x}$ . Then, an element  $u \in T\mathcal{M}$  is a vector tangent to the manifold in some point  $\mathbf{x} \in \mathcal{M}$ . Local coordinates for  $T\mathcal{M}$  are introduced by considering a local coordinate system  $(x^1, x^2, \dots, x^N)$  on  $\mathcal{M}$  and the components of  $y$  in such a coordinate system  $(y^1, y^2, \dots, y^N)$ . The  $2N$  numbers  $(x^1, x^2, \dots, x^N, y^1, y^2, \dots, y^N)$  constitute a local coordinate system on  $T\mathcal{M}$ . We can write synthetically  $u = (\mathbf{x}, \mathbf{y})$ .  $T\mathcal{M}$  is a  $2N$ -dimensional, differentiable manifold.

Let  $\pi$  be the mapping (*natural projection*)  $\pi : u = (\mathbf{x}, \mathbf{y}) \longrightarrow \mathbf{x}$ . ( $\mathbf{x} \in \mathcal{M}$ ,  $\mathbf{y} \in T\mathcal{M}_{\mathbf{x}}$ ). Then, the tern  $(T\mathcal{M}, \pi, \mathcal{M})$  is the *tangent bundle* to the base manifold  $\mathcal{M}$ . The image of the inverse mapping  $\pi^{-1}(\mathbf{x})$  is of course the tangent space  $T\mathcal{M}_{\mathbf{x}}$ , which is called the *fiber corresponding to the point  $\mathbf{x}$  in the fiber bundle*. One considers also sometimes the manifold  $\bar{T}\mathcal{M} = T\mathcal{M}/\{0\}$ , where 0 is the zero section of the projection  $\pi$ . We do not dwell further on the theory of the fiber bundles, and refer the reader to the wide and excellent literature on the subject [5].

The natural basis of the tangent space  $T_u(T\mathcal{M})$  at a point  $u = (\mathbf{x}, \mathbf{y}) \in T\mathcal{M}$  is  $\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^j} \right\}$ ,  $i, j = 1, 2, \dots, N$ .

A local coordinate transformation in the differentiable manifold  $T\mathcal{M}$  reads

$$\begin{cases} x'^i = x'^i(\mathbf{x}), & \det \left( \frac{\partial x'^i}{\partial x^j} \right) \neq 0, \\ y'^i = \frac{\partial x'^i}{\partial x^j} y^j. \end{cases} \quad (19)$$

Here,  $y^i$  is the *Liouville vector field* on  $T\mathcal{M}$ , i.e.  $y^i \frac{\partial}{\partial y^i}$ .

On account of Eq.(19), the natural basis of  $T\mathcal{M}_{\mathbf{x}}$  can be written as

$$\begin{cases} \frac{\partial}{\partial x^i} = \frac{\partial x'^k}{\partial x^i} \frac{\partial}{\partial x'^k} + \frac{\partial y'^k}{\partial x^i} \frac{\partial}{\partial y'^k}, \\ \frac{\partial}{\partial y^j} = \frac{\partial y'^k}{\partial y^j} \frac{\partial}{\partial y'^k}. \end{cases} \quad (20)$$

Second Eq.(20) shows therefore that the vector basis  $\left( \frac{\partial}{\partial y^j} \right)$ ,  $j = 1, 2, \dots, N$ , generates a distribution  $\mathcal{V}$  defined everywhere on  $T\mathcal{M}$  and integrable, too (*vertical distribution on  $T\mathcal{M}$* ).

If  $\mathcal{H}$  is a distribution on  $T\mathcal{M}$  supplementary to  $\mathcal{V}$ , namely

$$T_u(T\mathcal{M}) = \mathcal{H}_u \oplus \mathcal{V}_u \quad , \quad \forall u \in T\mathcal{M}, \quad (21)$$

then  $\mathcal{H}$  is called a *horizontal distribution*, or a *nonlinear connection* on  $T\mathcal{M}$ .

A basis for the distributions  $\mathcal{H}$  and  $\mathcal{V}$  are given respectively by  $\left\{ \frac{\delta}{\delta x^i} \right\}$  and



$\left\{ \frac{\partial}{\partial y^j} \right\}$ , where the basis in  $\mathcal{H}$  explicitly reads

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - H_i^j(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial y^j}. \quad (22)$$

Here,  $H_i^j(\mathbf{x}, \mathbf{y})$  are the *coefficients* of the nonlinear connection  $\mathcal{H}$ . The basis  $\left\{ \frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j} \right\} = \left\{ \delta_i, \dot{\partial}_j \right\}$  is called the *adapted basis*.

The dual basis to the adapted basis is  $\{dx^i, \delta y^j\}$ , with

$$\delta y^j = dy^j + H_i^j(\mathbf{x}, \mathbf{y}) dx^i. \quad (23)$$

A *distinguished tensor* (or *d-tensor*) *field of (r,s)-type* is a quantity whose components transform like a tensor under the first coordinate transformation (19) on  $T\mathcal{M}$  (namely they change as tensor in  $\mathcal{M}$ ). For instance, for a d-tensor of type (1,2):

$$R^i{}_{jk} = \frac{\partial x'^i}{\partial x^s} \frac{\partial x^r}{\partial x'^j} \frac{\partial x^p}{\partial x'^k} R^s{}_{rp}. \quad (24)$$

In particular, both  $\left\{ \frac{\delta}{\delta x^i} \right\}$  and  $\left\{ \frac{\partial}{\partial y^j} \right\}$  are d-(covariant) vectors, whereas  $\{dx^i\}$ ,  $\{\delta y^j\}$  are d-(contravariant) vectors.

A *generalized Lagrange space* is a pair  $\mathcal{GL}^N = (\mathcal{M}, g_{ij}(\mathbf{x}, \mathbf{y}))$ , with  $g_{ij}(\mathbf{x}, \mathbf{y})$  being a d-tensor of type (0,2) (covariant) on the manifold  $T\mathcal{M}$ , which is symmetric, non-degenerate<sup>5</sup> and of constant signature.

A function

$$L : (\mathbf{x}, \mathbf{y}) \in T\mathcal{M} \rightarrow L(\mathbf{x}, \mathbf{y}) \in \mathcal{R} \quad (25)$$

differentiable on  $\widehat{T\mathcal{M}}$  and continuous on the null section of  $\pi$  is named a *regular Lagrangian* if the Hessian of  $L$  with respect to the variables  $y^i$  is non-singular.

A generalized Lagrange space  $\mathcal{GL}^N = (\mathcal{M}, g_{ij}(\mathbf{x}, \mathbf{y}))$  is reducible to a *Lagrange space*  $\mathcal{L}^N$  if there is a regular Lagrangian  $L$  satisfying

$$g_{ij} = \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j} \quad (26)$$

on  $\widehat{T\mathcal{M}}$ . In order that  $\mathcal{GL}^N$  is reducible to a Lagrange space, a necessary condition is the total symmetry of the d-tensor  $\frac{\partial g_{ij}}{\partial y^k}$ . If such a condition is satisfied, and  $g_{ij}$  are 0-homogeneous in the variables  $y^i$ , then the function  $L = g_{ij}(\mathbf{x}, \mathbf{y}) y^i y^j$  is a solution of the system (26). In this case, the pair  $(\mathcal{M}, L)$  is a *Finsler space*<sup>6</sup>  $(\mathcal{M}, \Phi)$ , with  $\Phi^2 = L$ . One says that  $\mathcal{GL}^N$  is reducible to a Finsler space.

<sup>5</sup>Namely it must be  $\text{rank } \|g_{ij}(\mathbf{x}, \mathbf{y})\| = N$ .

<sup>6</sup>Let us recall that a Finsler space is a couple  $(\mathcal{M}, \Phi)$ , where  $\mathcal{M}$  is be an N-dimensional differential manifold and  $\Phi : T\mathcal{M} \Rightarrow \mathcal{R}$  a function  $\Phi(\mathbf{x}, \xi)$  defined for  $\mathbf{x} \in \mathcal{M}$  and  $\xi \in T_{\mathbf{x}}\mathcal{M}$  such that  $\Phi(\mathbf{x}, \cdot)$  is a possibly non symmetric norm on  $T_{\mathbf{x}}\mathcal{M}$ .

Of course,  $\mathcal{GL}^N$  reduces to a pseudo-Riemannian (or Riemannian) space  $(\mathcal{M}, g_{ij}(\mathbf{x}))$  if the d-tensor  $g_{ij}(\mathbf{x}, \mathbf{y})$  does not depend on  $\mathbf{y}$ . On the contrary, if  $g_{ij}(\mathbf{x}, \mathbf{y})$  depends only on  $\mathbf{y}$  (at least in preferred charts), it is a generalized Lagrange space which is locally Minkowskian.

Since, in general, a generalized Lagrange space is not reducible to a Lagrange one, it cannot be studied by means of the methods of symplectic geometry, on which — as is well known — analytical mechanics is based.

A linear  $\mathcal{H}$ -connection on  $T\mathcal{M}$  (or on  $\widehat{T\mathcal{M}}$ ) is defined by a couple of geometrical objects  $\mathcal{C}\Gamma(\mathcal{H}) = (L^i_{jk}, C^i_{jk})$  on  $T\mathcal{M}$  with different transformation properties under the coordinate transformation (19). Precisely,  $L^i_{jk}(\mathbf{x}, \mathbf{y})$  transform like the coefficients of a linear connection on  $\mathcal{M}$ , whereas  $C^i_{jk}(\mathbf{x}, \mathbf{y})$  transform like a d-tensor of type (1,2).  $\mathcal{C}\Gamma(\mathcal{H})$  is called *the metrical canonical  $\mathcal{H}$ -connection* of the generalized Lagrange space  $\mathcal{GL}^N$ .

In terms of  $L^i_{jk}$  and  $C^i_{jk}$  one can define two kinds of covariant derivatives: a *covariant horizontal (h-) derivative*, denoted by " $\mid$ ", and a *covariant vertical (v-) derivative*, denoted by " $\mid^s$ ". For instance, for the d-tensor  $g_{ij}(\mathbf{x}, \mathbf{y})$  one has

$$\begin{cases} g_{ij\mid k} = \frac{\delta g_{ij}}{\delta x^k} - g_{sj}L^s_{ik} - g_{is}L^s_{jk}; \\ g_{ij\mid^s k} = \frac{\partial g_{ij}}{\partial x^k} - g_{sj}C^s_{ik} - g_{is}C^s_{jk}. \end{cases} \quad (27)$$

The two derivatives  $g_{ij\mid k}$  and  $g_{ij\mid^s k}$  are both d-tensors of type (0,3).

The coefficients of  $\mathcal{C}\Gamma(\mathcal{H})$  can be expressed in terms of the following *generalized Christoffel symbols*:

$$\begin{cases} L^i_{jk} = \frac{1}{2}g^{is} \left( \frac{\delta g_{sj}}{\delta x^k} + \frac{\delta g_{ks}}{\delta x^j} + \frac{\delta g_{jk}}{\delta x^s} \right); \\ C^i_{jk} = \frac{1}{2}g^{is} \left( \frac{\partial g_{sj}}{\partial x^k} + \frac{\partial g_{ks}}{\partial x^j} + \frac{\partial g_{jk}}{\partial x^s} \right). \end{cases} \quad (28)$$

**Curvature and torsion in a generalized Lagrange space** By means of the connection  $\mathcal{C}\Gamma(\mathcal{H})$  it is possible to define a *d-curvature* in  $T\mathcal{M}$  by means of the tensors  $R^i_{j\mid kh}$ ,  $S^i_{j\mid kh}$  and  $P^i_{j\mid kh}$  given by

$$\begin{aligned} R^i_{j\mid kh} &= \frac{\delta L^i_{jk}}{\delta x^h} - \frac{\delta L^i_{jh}}{\delta x^k} + L^r_{jk}L^i_{rh} - L^r_{jh}L^i_{rk} + C^i_{jr}R^r_{kh}; \\ S^i_{j\mid kh} &= \frac{\partial C^i_{jk}}{\partial y^h} - \frac{\partial C^i_{jh}}{\partial y^k} + C^r_{jk}C^i_{rh} - C^r_{jh}C^i_{rk}; \\ P^i_{j\mid kh} &= \frac{\partial L^i_{jk}}{\partial y^h} - C^i_{jh} + C^i_{jr}P^r_{kh}. \end{aligned} \quad (29)$$

---

Notice that every Riemann manifold  $(\mathcal{M}, \mathbf{g})$  is also a Finsler space, the norm  $\Phi(\mathbf{x}, \xi)$  being the norm induced by the scalar product  $\mathbf{g}(\mathbf{x})$ .

A finite-dimensional Banach space is another simple example of Finsler space, where  $\Phi(\mathbf{x}, \xi) \equiv \|\xi\|$ .

Here, the d-tensor  $R_{jk}^i$  is related to the bracket of the basis  $\left\{ \frac{\delta}{\delta x^i} \right\}$  :

$$\left[ \frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j} \right] = R_{ij}^s \frac{\partial}{\partial y^s} \quad (30)$$

and is explicitly given by<sup>7</sup>

$$R_{jk}^i = \frac{\delta H_j^i}{\delta x^k} - \frac{\delta H_k^i}{\delta x^j}. \quad (31)$$

The tensor  $P_{jk}^i$ , together with  $T_{jk}^i$ ,  $S_{jk}^i$ , defined by

$$\begin{aligned} P_{jk}^i &= \frac{\partial H_j^i}{\partial y^k} - L_{jk}^i; \\ T_{jk}^i &= L_{jk}^i - L_{kj}^i; \\ S_{jk}^i &= C_{jk}^i - C_{kj}^i \end{aligned} \quad (32)$$

are the *d-tensors of torsion of the metrical connection*  $\mathcal{C}\Gamma(\mathcal{H})$ .

From the curvature tensors one can get the corresponding Ricci tensors of  $\mathcal{C}\Gamma(\mathcal{H})$ :

$$\begin{cases} R_{ij} = R_{ijs}^s; & S_{ij} = S_{ijs}^s; \\ \overset{1}{P}_{ij} = P_{ijs}^s & \overset{2}{P}_{ij} = P_{isj}^s, \end{cases} \quad (33)$$

and the scalar curvatures

$$R = g^{ij} R_{ij}; \quad S = g^{ij} S_{ij}. \quad (34)$$

Finally, the deflection *d-tensors associated to the connection*  $\mathcal{C}\Gamma(\mathcal{H})$  are

$$\begin{cases} D_j^i = y_{|j}^i = -H_j^i + y^s L_{sj}^i; \\ d_j^i = y_{|j}^i = \delta_j^i + y^s C_{sj}^i, \end{cases} \quad (35)$$

namely the h- and v-covariant derivatives of the Liouville vector fields.

In the generalized Lagrange space  $\mathcal{GL}^N$  it is possible to write the Einstein equations with respect to the canonical connection  $\mathcal{C}\Gamma(\mathcal{H})$  as follows:

$$\begin{cases} R_{ij} - \frac{1}{2} R g_{ij} = \kappa T_{ij}^H; & \overset{1}{P}_{ij} = \kappa T_{ij}^1; \\ S_{ij} - \frac{1}{2} S g_{ij} = \kappa T_{ij}^V; & \overset{2}{P}_{ij} = \kappa T_{ij}^2, \end{cases} \quad (36)$$

---

<sup>7</sup> $R_{jk}^i$  plays the role of a curvature tensor of the nonlinear connection  $\mathcal{H}$ . The corresponding tensor of torsion is instead

$$t_{jk}^i = \frac{\partial H_j^i}{\partial y^k} - \frac{\partial H_k^i}{\partial y^j}.$$

where  $\kappa$  is a constant and  $\overset{H}{T}_{ij}, \overset{V}{T}_{ij}, \overset{1}{T}_{ij}, \overset{2}{T}_{ij}$  are the components of the energy-momentum tensor.

### 3.2.2 Generalized Lagrangian Structure of $\widetilde{M}$

On the basis of the previous considerations, let us analyze the geometrical structure of the deformed Minkowski space of DSR  $\widetilde{M}$ , endowed with the by now familiar metric  $g_{\mu\nu,DSR}(E)$ . As said in Sect.2,  $E$  is the energy of the process measured by the detectors in Minkowskian conditions. Therefore,  $E$  is a function of the velocity components,  $u^\mu = dx^\mu/d\tau$ , where  $\tau$  is the (Minkowskian) proper time<sup>8</sup>:

$$E = E\left(\frac{dx^\mu}{d\tau}\right). \quad (37)$$

The derivatives  $dx^\mu/d\tau$  define a contravariant vector tangent to  $M$  at  $x$ , namely they belong to  $TM_{\mathbf{x}}$ . We shall denote this vector (according to the notation of the previous Subsubsection) by  $\mathbf{y} = (y^\mu)$ . Then,  $(\mathbf{x}, \mathbf{y})$  is a point of the tangent bundle to  $M$ . We can therefore consider the generalized Lagrange space  $\mathcal{GL}^4 = (M, g_{\mu\nu}(\mathbf{x}, \mathbf{y}))$ , with

$$\begin{cases} g_{\mu\nu}(\mathbf{x}, \mathbf{y}) = g_{\mu\nu,DSR}(E(\mathbf{x}, \mathbf{y})), \\ E(\mathbf{x}, \mathbf{y}) = E(\mathbf{y}). \end{cases} \quad (38)$$

Then, it is possible to prove the following theorem [6]:

*The pair  $\mathcal{GL}^4 = (M, g_{DSR,\mu\nu}(\mathbf{x}, \mathbf{y})) \equiv \widetilde{M}$  is a generalized Lagrange space which is not reducible to a Riemann space, or to a Finsler space, or to a Lagrange space.*

Notice that such a result is strictly related to the fact that the deformed metric tensor of DSR is diagonal.

If an external electromagnetic field  $F_{\mu\nu}$  is present in the Minkowski space  $M$ , in  $\widetilde{M}$  the deformed electromagnetic field is given by  $\widetilde{F}_\nu^\mu(\mathbf{x}, \mathbf{y}) = g_{DSR}^{\mu\rho} F_{\rho\nu}(\mathbf{x})$  (see Eq.(8)). Such a field is a d-tensor and is called *the electromagnetic tensor of the generalized Lagrange space*. Then, the nonlinear connection  $\mathcal{H}$  is given by

$$H_\nu^\mu = \left\{ \begin{matrix} \mu \\ \nu\rho \end{matrix} \right\} y^\rho - \widetilde{F}_\nu^\mu(\mathbf{x}, \mathbf{y}), \quad (39)$$

where  $\left\{ \begin{matrix} \mu \\ \nu\rho \end{matrix} \right\}$ , the Christoffel symbols of the Minkowski metric  $g_{\mu\nu}$ , are zero, so that

$$H_\nu^\mu = -\widetilde{F}_\nu^\mu(\mathbf{x}, \mathbf{y}), \quad (40)$$

namely, *the connection coincides with the deformed field*.

---

<sup>8</sup>Contrarily to ref.[6], we shall not consider the restrictive case of a classical (non-relativistic) expression of the energy, but assume a general dependence of  $E$  on the velocity (eq.(38)).

The adapted basis of the distribution  $\mathcal{H}$  reads therefore

$$\frac{\delta}{\delta x^\mu} = \frac{\partial}{\partial x^\mu} + \tilde{F}_\mu^\nu(\mathbf{x}, \mathbf{y}) \frac{\partial}{\partial y^\nu}. \quad (41)$$

The local covector field of the dual basis (cfr. Eq.(23)) is given by

$$\delta y^\mu = dy^\mu - \tilde{F}_\nu^\mu(\mathbf{x}, \mathbf{y}) dx^\nu. \quad (42)$$

### 3.2.3 Canonical Metric Connection of $\widetilde{M}$

The derivation operators applied to the deformed metric tensor of the space  $\mathcal{GL}^4 = \widetilde{M}$  yield

$$\frac{\delta g_{DSR\mu\nu}}{\delta x^\rho} = \frac{\partial g_{DSR\mu\nu}}{\partial x^\rho} + \tilde{F}_\rho^\sigma \frac{\partial g_{DSR\mu\nu}}{\partial y^\sigma} = \tilde{F}_\rho^\sigma \frac{\partial g_{DSR\mu\nu}}{\partial E} \frac{\partial E}{\partial y^\sigma}, \quad (43)$$

$$\frac{\partial g_{DSR\mu\nu}}{\partial y^\sigma} = \frac{\partial g_{DSR\mu\nu}}{\partial E} \frac{\partial E}{\partial y^\sigma}. \quad (44)$$

Then, the coefficients of the canonical metric connection  $\mathcal{C}\Gamma(\mathcal{H})$  in  $\widetilde{M}$  (see Eq.(28)) are given by

$$\begin{cases} L_{\nu\rho}^\mu = \frac{1}{2} g_{DSR}^{\mu\sigma} \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\sigma\nu}}{\partial E} \tilde{F}_\rho^\alpha + \frac{\partial g_{DSR\sigma\rho}}{\partial E} \tilde{F}_\nu^\alpha - \frac{\partial g_{DSR\nu\rho}}{\partial E} \tilde{F}_\sigma^\alpha \right), \\ C_{\nu\rho}^\mu = \frac{1}{2} g_{DSR}^{\mu\sigma} \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\sigma\nu}}{\partial E} \delta_\rho^\alpha + \frac{\partial g_{DSR\sigma\rho}}{\partial E} \delta_\nu^\alpha - \frac{\partial g_{DSR\nu\rho}}{\partial E} \delta_\sigma^\alpha \right). \end{cases} \quad (45)$$

The vanishing of the electromagnetic field tensor,  $F_\rho^\alpha = 0$ , implies  $L_{\nu\rho}^\mu = 0$ .

One can define the deflection tensors associated to the metric connection  $\mathcal{C}\Gamma(\mathcal{H})$  as follows (cfr. Eq.(36)):

$$\begin{aligned} D_\nu^\mu &= y_{|\nu}^\mu = \frac{\delta y^\mu}{\delta x^\nu} + y^\alpha L_{\alpha\nu}^\mu = \tilde{F}_\nu^\mu + y^\alpha L_{\alpha\nu}^\mu; \\ d_\nu^\mu &= y_{|\nu}^\mu = \delta_\nu^\mu + y^\alpha C_{\alpha\nu}^\mu. \end{aligned} \quad (46)$$

The covariant components of these tensors read

$$\begin{aligned} D_{\mu\nu} &= g_{\mu\sigma, DSR} D_\nu^\sigma = g_{\mu\sigma, DSR} \left( \tilde{F}_\nu^\sigma + y^\alpha L_{\alpha\nu}^\sigma \right) = \\ &= F_{\mu\nu}(\mathbf{x}) + \frac{1}{2} y^\sigma \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\mu\sigma}}{\partial E} \tilde{F}_\nu^\alpha + \frac{\partial g_{DSR\mu\nu}}{\partial E} \tilde{F}_\sigma^\alpha - \frac{\partial g_{DSR\sigma\nu}}{\partial E} \tilde{F}_\mu^\alpha \right); \\ d_{\mu\nu} &= g_{\mu\sigma, DSR} d_\nu^\sigma = \\ &= g_{DSR, \mu\nu} + \frac{1}{2} y^\sigma \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\mu\sigma}}{\partial E} \delta_\nu^\alpha + \frac{\partial g_{DSR\mu\nu}}{\partial E} \delta_\sigma^\alpha - \frac{\partial g_{DSR\sigma\nu}}{\partial E} \delta_\mu^\alpha \right). \end{aligned} \quad (47)$$

It is important to stress explicitly that, on the basis of the results of 3.2.1, *the deformed Minkowski space  $\widetilde{M}$  does possess curvature and torsion*, namely it is endowed with a very rich geometrical structure. This permits to understand the variety of new physical phenomena that occur in it (as compared to the standard Minkowski space) [1, 2].

Following ref.[6], let us show how the formalism of the generalized Lagrange space allows one to recover some results on the phenomenological energy-dependent metrics discussed in Sect.2.

Consider the following metric ( $c = 1$ ):

$$ds^2 = a(E)dt^2 + (dx^2 + dy^2 + dz^2) \quad (48)$$

where  $a(E)$  is an arbitrary function of the energy and spatial isotropy ( $b^2 = 1$ ) has been assumed. In absence of an external electromagnetic field ( $F_{\mu\nu} = 0$ ), the non-vanishing components  $C^\mu_{\nu\rho}$  of the canonical metric connection  $\mathcal{C}\Gamma(\mathcal{H})$  (see Eq.(46)) are

$$\left\{ \begin{array}{l} C^0_{00} = \frac{a'}{a}y^0, \quad C^0_{01} = -\frac{a'}{a}y^1, \quad C^0_{02} = -\frac{a'}{a}y^2, \quad C^0_{03} = \frac{a'}{a}y^3, \\ \\ C^1_{00} = -a'y^1, \quad C^2_{00} = -a'y^2, \quad C^3_{00} = -a'y^3, \end{array} \right. \quad (49)$$

where the prime denotes derivative with respect to  $E$ :  $a' = \frac{da}{dE}$ .

According to the formalism of generalized Lagrange spaces, we can write the Einstein equations in vacuum corresponding to the metrical connection of the deformed Minkowski space (see Eqs.(37)). It is easy to see that the independent equations are given by

$$a' = 0; \quad (50)$$

$$2aa'' - (a')^2 = 0. \quad (51)$$

The first equation has the solution  $a = \text{const.}$ , namely we get the Minkowski metric. Eq.(52) has the solution

$$a(E) = \frac{1}{4} \left( a_0 + \frac{E}{E_0} \right)^2, \quad (52)$$

where  $a_0$  and  $E_0$  are two integration constants.

This solution represents the time coefficient of an over-Minkowskian metric. For  $a_0 = 0$  it coincides with (the time coefficient of) the phenomenological metric of the strong interaction, Eq.(6). On the other hand, by choosing  $a_0 = 1$ , one gets the time coefficient of the metric for gravitational interaction, Eq.(6').

In other words, *considering  $\widetilde{M}$  as a generalized Lagrange space permits to recover (at least partially) the metrics of two interactions (strong and gravitational) derived on a phenomenological basis.*

It is also worth noticing that this result shows that *a spacetime deformation (of over-Minkowskian type) exists even in absence of an external electromagnetic field* (remember that Eqs.(51),(52) have been derived by assuming  $F_{\mu\nu} = 0$ ).

### 3.2.4 Intrinsic Physical Structure of a Deformed Minkowski Space: Gauge Fields

As we have seen, the deformed Minkowski space  $\widetilde{M}$ , considered as a generalized Lagrange space, is endowed with a rich geometrical structure. But the important point, to our purposes, is the presence of a physical richness, intrinsic to  $\widetilde{M}$ . Indeed, let us introduce the following *internal electromagnetic field tensors* on  $\mathcal{GL}^4 = \widetilde{M}$ , defined in terms of the deflection tensors:

$$\begin{aligned}\mathcal{F}_{\mu\nu} &\equiv \frac{1}{2}(D_{\mu\nu} - D_{\nu\mu}) = \\ &= F_{\mu\nu}(\mathbf{x}) + \frac{1}{2}y^\sigma \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\mu\sigma}}{\partial E} \widetilde{F}_\nu^\alpha - \frac{\partial g_{DSR\nu\sigma}}{\partial E} \widetilde{F}_\mu^\alpha \right)\end{aligned}\quad (53)$$

(horizontal electromagnetic internal tensor) and

$$\begin{aligned}f_{\mu\nu} &\equiv \frac{1}{2}(d_{\mu\nu} - d_{\nu\mu}) = \\ &= \frac{1}{2}y^\sigma \frac{\partial E}{\partial y^\alpha} \left( \frac{\partial g_{DSR\mu\sigma}}{\partial E} \delta_\nu^\alpha - \frac{\partial g_{DSR\nu\sigma}}{\partial E} \delta_\mu^\alpha \right)\end{aligned}\quad (54)$$

(vertical electromagnetic internal tensor).

The internal electromagnetic h- and v-fields  $\mathcal{F}_{\mu\nu}$  and  $f_{\mu\nu}$  satisfy the following *generalized Maxwell equations*

$$\begin{aligned}2(\mathcal{F}_{\mu\nu|\rho} + \mathcal{F}_{\nu\rho|\mu} + \mathcal{F}_{\rho\mu|\nu}) &= y^\alpha (R_{\mu\nu}^\beta C_{\beta\alpha\rho} + R_{\nu\rho}^\beta C_{\beta\alpha\mu} + R_{\rho\mu}^\beta C_{\beta\alpha\nu}), \\ R_{\mu\nu}^\beta &= g^{\beta\sigma} \frac{\partial \mathcal{F}_{\mu\nu}}{\partial x^\sigma};\end{aligned}\quad (55)$$

$$\mathcal{F}_{\mu\nu|\rho} + \mathcal{F}_{\nu\rho|\mu} + \mathcal{F}_{\rho\mu|\nu} = f_{\mu\nu|\rho} + f_{\nu\rho|\mu} + f_{\rho\mu|\nu}; \quad (56)$$

$$f_{\mu\nu|\rho} + f_{\nu\rho|\mu} + f_{\rho\mu|\nu} = 0. \quad (57)$$

Let us stress explicitly the different nature of the two internal electromagnetic fields. In fact, the horizontal field  $\mathcal{F}_{\mu\nu}$  is strictly related to the presence of the external electromagnetic field  $F_{\mu\nu}$ , and vanishes if  $F_{\mu\nu} = 0$ . On the contrary, *the vertical field  $f_{\mu\nu}$  has a geometrical origin, and depends only on the deformed metric tensor  $g_{DSR\mu\nu}(E(\mathbf{y}))$  of  $\mathcal{GL}^4 = \widetilde{M}$  and on  $E(\mathbf{y})$ . Therefore, it is present also in space-time regions where no external electromagnetic field occurs.* As we shall see, this fact has deep physical implications.

A few remarks are in order. First, the main results obtained for the (abelian) electromagnetic field can be probably generalized (with suitable changes) to non-abelian gauge fields. Second, the presence of the internal electromagnetic h- and v-fields  $\mathcal{F}_{\mu\nu}$  and  $f_{\mu\nu}$ , intrinsic to the geometrical structure of  $\widetilde{M}$  as a generalized

Lagrange space, is the cornerstone to build up a *dynamics (of merely geometrical origin) internal to the deformed Minkowski space*.

The important point worth emphasizing is that *such an intrinsic dynamics springs from gauge fields*. Indeed, the two internal fields  $\mathcal{F}_{\mu\nu}$  and  $f_{\mu\nu}$  (in particular the latter one) do satisfy equations of the gauge type (cfr. Eqs.(57)-(58)). Then, we can conclude that *the (energy-dependent) deformation of the metric of  $\widetilde{M}$ , which induces its geometrical structure as generalized Lagrange space, leads in turn to the appearance of (internal) gauge fields*.

Such a fundamental result can be schematized as follows:

$$\widetilde{M} = (M, g_{DSR\mu\nu}(E)) \implies \mathcal{GL}^4 = (M, g_{\mu\nu}(\mathbf{x}, \mathbf{y})) \implies (\widetilde{M}, \mathcal{F}_{\mu\nu}, f_{\mu\nu}) \quad (58)$$

(with self-explanatory meaning of the notation).

We want also to stress explicitly that this result follows by the fact that, in deforming the metric of the space-time, *we assumed the energy as the physical (non-metric) observable on which letting the metric coefficients depend*. This is crucial in stating the generalized Lagrangian structure of  $\widetilde{M}$ , as shown above.

### 3.3 Possible evidence for DSR internal gauge fields: Shadow of light

We want now to discuss some results on anomalous interference effects, which admit a quite straightforward interpretation in terms of the intrinsic gauge fields of DSR.

In double-slit-like experiments in the infrared range, we collected evidences of an anomalous behaviour of photon systems under particular (energy and space) constraints [7, 8, 9, 10]. The experimental set-up is reported in Fig.1.

This layout shows the horizontal view of the interior of a closed box divided into different rooms by panels. The box was 20 *cm* long, 12 *cm* large and 7 *cm* high. It contained two infrared LEDs  $S_1$  and  $S_2$ , three detectors A, B and C (either photodiodes or phototransistors) and three apertures  $F_1$ ,  $F_2$  and  $F_3$ . The source  $S_1$  was aligned with the detector A through the aperture  $F_1$ , the source  $S_2$  was aligned with the detector C which was right on the aperture  $F_3$ . The detector B was in front of the aperture  $F_2$  and did not receive any photon directly. The position of the detectors, the sources and the apertures was designed so that the detector A was not influenced by the lighting state of the source  $S_2$  according to the laws of physics governing photons propagation. In other words, A did not have to distinguish whether  $S_2$  was on or off. Besides, in order to prevent reflections of photons, the internal surfaces of the box had been coated by an absorbing material. While the detectors B and C were controlling detectors, A was devoted to perform the actual experiment. In particular, we compared the signal, measured on A when  $S_1$  was on and  $S_2$  was off, with the signal on A when both sources  $S_1$  and  $S_2$  were on. As to what it has been said about the incapability of A to distinguish between  $S_2$  off or on, these two compared conditions were expected to produce compatible results. However, it turned out that the sampling of the signal on A with  $S_1$  on and  $S_2$  on and the



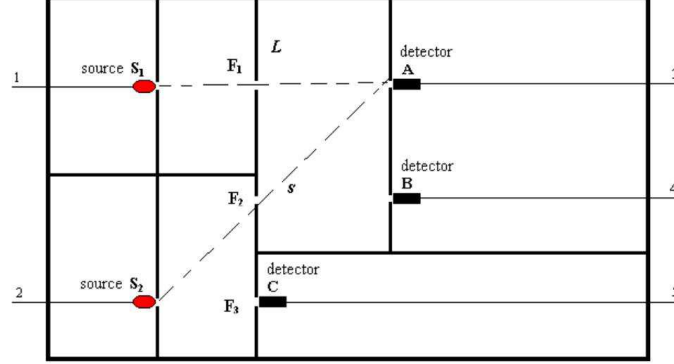


Figure 1: Schematic layout of the box used to detect the anomalous interference effect.

sampling of the signal on A when only  $S_1$  was on do not belong to the same population and are represented by two different gaussian distributions whose mean values are significantly different. Besides, the difference between the two mean values was less than  $4.5 \mu eV$ , as predicted by the theory of Deformed Space-time [1, 2]. Since it was experimentally verified that no photons passed through the aperture  $F_2$ , this result shows an anomalous behaviour of the photon system. The same experiment was carried out by different sources, detectors, by two different boxes and different measuring systems. Nevertheless, every time we obtained the same anomalous result [8, 9, 10]. Moreover, the same kind of geometrical structure and the same spatial distances were used in other kind of experiments carried out in the microwave region of the spectrum and by a laser system [11, 12, 1, 2]. Although these experiments had completely different experimental set-ups from our initial one, they succeeded in finding out the same kind of anomalous behaviour that we had found out by the box experiments.

The anomalous effect in photon systems, at least in those experimental set-ups that were used, disagrees both with standard quantum mechanics (Copenhagen interpretation) and with classical and quantum electrodynamics. Some possible interpretations can be given in terms of either the existence of de Broglie-Bohm pilot waves associated to photons, and/or the breakdown of local Lorentz invariance (LLI) [7, 8, 9, 10]. Besides, it turns out that it is also possible to move a step forward and hypothesise the existence of an intriguing connection between the pilot wave interpretation and that involving LLI breakdown. One might assume that the pilot wave is, in the framework of LLI breakdown, a local deformation of the flat Minkowskian spacetime.

The interpretation in terms of DSR is quite straightforward. Under the energy threshold  $E_{0,em}=4.5 \mu eV$ , the metric of the electromagnetic interaction

is no longer Minkowskian. The corresponding space-time is deformed. Such a space-time deformation shows up as the hollow wave accompanying the photon, and is able to affect the motion of other photons. This is the origin of the anomalous interference observed (*shadow of light*). The difference of signal measured by the detector A in all the double-slit experiments can be regarded as the energy spent to deform space-time. In space regions where the external electromagnetic field is present (regions of "standard" photon behavior), we can associate such energy to the difference  $\Delta\mathcal{E}$ , Eq.(16), between the energy density corresponding to the external e.m. field  $F_{\mu\nu}$  and that of the deformed one  $\tilde{F}_{\mu\nu}$  given by Eq.(8).

But it is known from the experimental results that the anomalous interference effects observed can be explained in terms of the shadow of light, namely in terms of the hollow waves present in space regions where no external e.m. field occurs. How to account for this anomalous photon behavior within DSR? The answer is provided by the internal structure of the deformed Minkowski space discussed above. In fact, we have seen that the structure of the deformed Minkowski space  $\tilde{M}$  as Generalized Lagrange Space implies the presence of two internal e.m. fields, the horizontal field  $\mathcal{F}_{\mu\nu}$  and the vertical one,  $f_{\mu\nu}$ . Whereas  $\mathcal{F}_{\mu\nu}$  is strictly related to the presence of the external electromagnetic field  $F_{\mu\nu}$ , vanishing if  $F_{\mu\nu} = 0$ , the vertical field  $f_{\mu\nu}$  is geometrical in nature, depending only on the deformed metric tensor  $g_{DSR,\mu\nu}(E)$  of  $GL^4 = \tilde{M}$  and on  $E$ . Therefore, it is present also in space-time regions where no external electromagnetic field occurs. In our opinion, the arising of the internal electromagnetic fields associated to the deformed metric of  $\tilde{M}$  as Generalized Lagrange space is at the very physical, *dynamic* interpretation of the experimental results on the anomalous photon behavior. Namely, *the dynamic effects of the hollow wave of the photon, associated to the deformation of space-time* — which manifest themselves in the photon behavior contradicting both classical and quantum electrodynamics —, *arise from the presence of the internal v-electromagnetic field  $f_{\mu\nu}$*  (in turn strictly connected to the geometrical structure of  $\tilde{M}$ ).

Moreover, as is well known, in relativistic theories, the vacuum is nothing but Minkowski geometry. A LLI breaking connected to a deformation of the Minkowski space is therefore associated to a lack of Lorentz invariance of the vacuum. Then, the view by Kostelecky [13] that the breakdown of LLI is related to the lack of Lorentz symmetry of the vacuum accords with our results in the framework of DSR, provided that the quantum vacuum is replaced by the geometric vacuum.

## 4 Conclusions and perspectives

As is well known, successfully embodying gauge fields in a space-time structure is one of the basic goals of the research in theoretical physics starting from the beginning of the XX century. The almost unique tool to achieve such objective is increasing the number of space-time dimensions. In such a kind of theories (whose prototype is the celebrated Kaluza-Klein formalism), one

preserves the usual (special-relativistic or general-relativistic) structure of the four-dimensional space-time, and gets rid of the non-observable extra dimensions by compactifying them (for example to circles). Then the motions of the extra metric components over the standard Minkowski space satisfy identical equations to gauge fields. The gauge invariance of these fields is simply a consequence of the Lorentz invariance in the enlarged space. In this framework, gauge fields are *external* to the space-time, because they are *added* to it by the hypothesis of extra dimensions.

In the case of the DSR theory, gauge fields arise from the very geometrical, basic structure of  $\widetilde{M}$ , namely they are a consequence of the metric deformation. The arising gauge fields are *intrinsic and internal to the deformed space-time, and do not need to be added from the outside*. As a matter of fact, *DSR is the first theory based on a four-dimensional space-time able to embody gauge fields in a natural way*.

Such a conventional, intrinsic gauge structure is related to a *given* deformed Minkowski space  $\widetilde{M}$ , in which the deformed metric is fixed:

$$\widetilde{M} = (M, \bar{g}_{DSR\mu\nu}(E)). \quad (59)$$

On the contrary, with varying  $g_{DSR}$ , we have another gauge-like structure — as already stressed in Sect.3 — namely what we called an external metric gauge. In the latter case, the gauge freedom amounts to choosing the metric according to the interaction considered.

The circumstance that the deformed Minkowski space  $\widetilde{M}$  is endowed with the geometry of a generalized Lagrange space testifies the richness of non-trivial mathematical properties present in the seemingly so simple structure of the deformation of the Minkowski metric. In this connection, let us recall that  $\widetilde{M}$  (contrarily to the usual Minkowski space) is not flat, but does possess curvature and torsion (see 3.2.1).

Let us stress that — as already mentioned — the deformed Minkowski space  $\widetilde{M}$  can be naturally embedded in a five-dimensional Riemannian space  $\mathfrak{R}_5$  (see [1, 2]). We denoted by DR5 the generalized theory based on this five-dimensional space.

In embedding the deformed Minkowski space  $\widetilde{M}$  in  $\mathfrak{R}_5$ , *energy does lose its character of dynamic parameter* (the role it plays in DSR), *by taking instead that of a true metrical coordinate,  $E = x^5$* , on the same footing of the space-time ones. This has a number of basic implications. In such a change of role of energy, with the consequent passage from  $\widetilde{M}$  to  $\mathfrak{R}_5$ , some of the geometrical and dynamic features of DSR are lost, whereas others are still present and new properties appear. The first one is of geometrical nature, and is just the passage from a (flat) pseudoeuclidean metric to a genuine (curved) Riemannian one. The other consequences pertain to both symmetries and dynamics. Among the former, we recall the basic one — valid at the slicing level  $x^5 = \text{const.}$  ( $dx^5 = 0$ ) —, related to the Generalized Lagrange Space structure of  $\widetilde{M}$ , which implies *the natural arising of gauge fields*, intimately related to the inner geometry of the deformed Minkowski space. Let us also stress that, in the framework of  $\mathfrak{R}_5$ , the dependence

of the metric coefficients on a true metric coordinate make them fully analogous to the gauge functions of non-abelian gauge theories, thus implementing DR5 as a metric gauge theory (in the sense specified in Subsect.3.1). Let us recall that the metric homomorphisms of  $\mathfrak{R}_5$  are strictly connected to the invariance under what we called the Metric Gaugement Process of DSR (see Subsect.3.1).

Concerning the influence of the extra dimension on the physics in the four-dimensional deformed space-time, points worth investigating are the possible connection between Lorentz invariance in DR5 and the usual gauge invariance, and the occurrence of parity violation as consequence of space anisotropy when viewed from the standpoint of the space-time-energy manifold  $\mathfrak{R}_5$ .

A further basic topic deserving study in DSR is the extension to the non-abelian case of the results obtained for the abelian gauge fields (like the e.m. one), based on the structure of the deformed Minkowski space  $\widehat{M}$  as Generalized Lagrange Space (see Subsubsect.3.2.1). In other words, it would be worth verifying if also non-abelian internal gauge fields can exist in absence of external fields, due to the intrinsic geometry of  $M$ .

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